Attitude Instability in Steady Rolling and Roll Resonance

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It has been pointed out in an earlier paper that unsatisfactory response characteristics in steady-rolling maneuvers are not necessarily related to divergence in linear approximation, and it is the aircraft's angular position that experiences large deviations in the early stages of the maneuver. Two complementary causes of this phenomenon are now considered. It is shown that steady rolling results in unstable angular positions of the aircraft without regard to the value of the rolling velocity. The instability, however, is gentle and not critical by itself; it may nevertheless provide a propicious ground for perturbing actions, particularly the resonance effect. This effect, though apparent only in second-order approximation, induces large deviations of the response in pitch angle in the first 5-10 sec, and is essential in determining unsatisfactory response characteristics. It is shown that the main parameters that control attitude instability are the lift curve and particularly the side-force curve slopes: low values promote the occurence of a rough resonance affecting the pitch angle at certain rolling velocities which would not develop for some higher values of these parameters. Resonance values of the roll rate mainly depend on the longitudinal and the directional static stability and are higher the greater the static stability.

Nomenclature†

A	= matrix of the fifth-order linear system in β,α,q,r, and p
B	= matrix of the fourth-order linear system in β,α,q, and r
g	= gravity acceleration
h	= shorthand notation for $(-1/y_\beta + 1/z_\alpha)$
I	= unit matrix
I_x, I_y, I_z	= mass moments of inertia about principal axes
$i_1 = (I_z - I_y)/I_x$, $i_2 = (I_z - I_x)/I_y$, $i_3 = (I_y - I_x)/I_z$	= nondimensional inertia coefficients
k	= shorthand notation for $(-1/y_\beta - 1/z_\alpha)$
l	= rolling moment per I_x
m	= pitching moment per I_y
n	= yawing moment per I_z
p,q,r	= scalar components with respect to the principal axes of the aircraft angular velocity
t	= time
V	= velocity of the aircraft center of mass
W	= aircraft weight
y	= side force over aircraft mass and speed
z	= aerodynamic force along z principal axis over mass and speed
α	= incidence
β	= angle of sideslip
$\epsilon = g/V$	= small parameter
ζ	= rudder deflection angle
η	= elevator deflection angle
θ	= elevation angle of the x principal axis
ξ	= aileron deflection angle
σ	= real part of an eigenvalue of matrix A or B
ϕ	= lateral attitude angle
ω	= imaginary part of an eigenvalue of matrix A or B

Subscripts

0	= denotes approximation of order 0 with respect to ϵ
1	= denotes first approximation with respect to ϵ
2	= denotes second approximation with respect to ϵ
$p,q,r,\alpha,\beta,\xi,\eta,\zeta,\dot\alpha$	= denote partial derivative due to the respective quantity (e.g., $y_\beta = \partial y/\partial\beta$, $m_{\dot\alpha} = \partial m/\partial(\mathrm{d}\alpha/\mathrm{d}t)$, $l_{\xi\alpha}\partial^2 l/\partial\xi\partial\alpha$)

Superscript

T	= transpose

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†Lower case and capital Greek deltas (δ and Δ) denote deviations from some values of the system variables defined by given laws of variation; a dot over a symbol stands for total time derivative.

Introduction

IT was generally accepted since Phillips' now classical note[1] published in 1948 that, with positive static stability and positive damping, the only criterion necessary to consider when studying the flying qualities in steady rolling is that corresponding to the sign of the free term of the characteristic polynomial of the simplified fifth- or fourth-order linear equations system (in β,α,q,r, and p as the fifth variable) obtained by neglecting the gravity terms (see Ref. 2, p. 553). Negative free term means divergence, which was held accountable for the high-peak-amplitude deviations of some of the system variables, particularly sideslip and incidence. It is shown, however,[3] that divergence connected with the negative free term of the characteristic polynomial is not necessarily critical. Response over a reasonably long interval may be quite satisfactory with a negative free term while it is also possible that the response may be unsatisfactory with a positive free term. It is also shown that the attitude angles are the first to take on dangerous values, which eventually may induce unsatisfactory behavior of sideslip and incidence.

The calculated time histories of the attitude θ and the incidence, as compared to the ideal variation of these angles, are plotted in Figs. 1 and 2 in some seemingly contradictory situations for the aircraft and flight condition used as an example in Ref. 3. Diagrams in Fig. 1 represent (in full lines) the response characteristics in α and θ in a rolling motion for which the conventional simplified linear analysis predicts unsatisfactory behavior.‡ In Fig. 2 the variation of α and θ in a

‡According to the criterion related to the sign of the free term of the characteristic polynomial, unsatisfactory behavior is predicted for roll rate values in the range from 3.24-4.41 rad/sec for the illustrative example used.

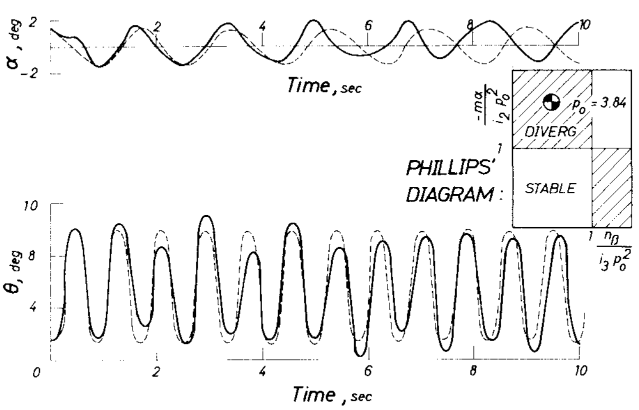

Fig. 1 Simulated and ideal variation of α and θ with t in steady rolling maneuver, divergent according to conventional linear analysis ($p_0=3.84$ rad/sec).

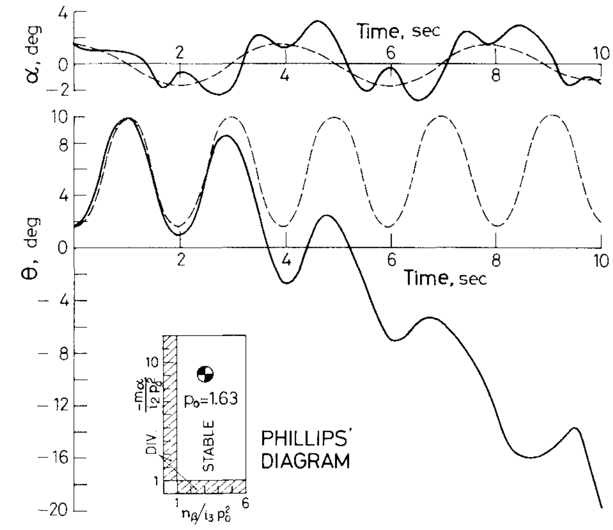

Fig. 2 Simulated and ideal variation of α and θ with t in a steady rolling maneuver, stable according to conventional linear analysis ($p_0 = 1.63$ rad/sec).

rolling motion that corresponds to a situation well within the stable domain according to the conventional linear analysis is plotted.

To explain the unexpected violent deviation experienced by the position angle θ in the latter situation, two complementary causes will be considered: a) instability with respect to the attitude angles ϕ and θ considered as additional state variables of the system, and b) the roll resonance.

Some Earlier Results

Generating System and Generating Solution

The curves in full lines in Figs. 1 and 2 represent the time histories of α and θ calculated by integrating a fairly complete nonlinear system, i.e.,

$$\dot\beta = y_\beta\beta + p\alpha - r + (g/V)\sin\phi\cos\theta \tag{1a}$$

$$\dot\alpha = z_\alpha\alpha + q - p\beta + (g/V)\cos\phi\cos\theta \tag{1b}$$

$$\dot q = \bar m_\alpha\alpha + \bar m_q q - m_{\dot\alpha}p\beta + i_2 pr + (g/V)m_{\dot\alpha}\cos\phi\cos\theta + m_\eta\eta \tag{1c}$$

$$\dot r = n_\beta\beta + n_{\xi\alpha}\xi\alpha + n_q q + n_r r + n_p p + n_{p\alpha}p\alpha - i_3 pq + n_\xi\xi + n_\zeta\zeta \tag{1d}$$

$$\dot p = l_\beta\beta + l_{\xi\alpha}\xi\alpha + l_q q + l_r r + l_p p + l_{\beta\alpha}\beta\alpha + l_{r\alpha}r\alpha - i_1 qr + l_\xi\xi + l_\zeta\zeta \tag{1e}$$

$$\dot\phi = p + q\sin\phi\tan\theta + r\cos\phi\tan\theta \tag{1f}$$

$$\dot\theta = q\cos\phi - r\sin\phi \tag{1g}$$

with $\bar m_\alpha = m_\alpha + z_\alpha m_{\dot\alpha}, \bar m_q = m_q + m_{\dot\alpha}$.[3,4] They are compared with an ideal periodic variation of the respective variables, when the roll rate is constant $(p=p_0)$, (dashed lines), namely

$$\alpha = -(g/Vz_\alpha)\cos p_0 t$$

$$\theta = (g/2V)\,[-(1/y_\beta + 1/z_\alpha) + (1/y_\beta - 1/z_\alpha)\cos 2p_0 t]$$

for the same initial conditions and control input [see Eqs. (6)].

The ideal solution is obtained as a first approximation with respect to the small parameter $\epsilon = g/V$. The method is described in Ref. 3 and will be outlined in the following.

Ratio g/V, and implicitly the gravitational terms in system (1) are small as compared to most of the coefficients. This gives rise to the idea of seeking the solution in the form of power series in g/V, i.e., if $\epsilon = g/V$, in the form

$$\beta = \beta_0 + \epsilon\beta_1 + \epsilon^2\beta_2 + \dots \qquad \alpha = \alpha_0 + \epsilon\alpha_1 + \epsilon^2\alpha_2 + \dots \tag{2a,b}$$

$$q = q_0 + \epsilon q_1 + \epsilon^2 q_2 + \dots \qquad r = r_0 + \epsilon r_1 + \epsilon^2 r_2 + \dots \tag{2,c,d}$$

$$p = p_0 + \epsilon p_1 + \epsilon^2 p_2 + \dots \qquad \phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \dots \tag{2e,f}$$

$$\theta = \theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + \dots \tag{2g}$$

The control functions used are of the form

$$\xi = \xi_0 + \epsilon\xi_1 \qquad \eta = \eta_0 + \epsilon\eta_1 \qquad \zeta = \epsilon_0 + \epsilon\zeta_1$$

If

$$\xi_0 = p_0(-l_p n_\zeta + n_p l_\zeta)/\delta_0, \quad \eta_0 = 0,$$
$$\zeta_0 = p_0(l_p n_\xi - n_p l_\xi)/\delta_0, \quad \delta_0 = l_\xi n_\zeta - n_\xi l_\zeta \tag{3}$$

with $p_0 = \text{const}$, we have $\beta_0 = \alpha_0 = q_0 = r_0 = \theta_0 = 0$ and $\phi_0 = p_0 t$. Then by transforming Eq. (1) according to Eqs. (2) and (3), and dividing each of the transformed equations by ϵ, we obtain

$$[\dot\beta_1, \dot\alpha_1, \dot q_1, \dot r_1, \dot p_1]^T = A[\beta_1, \alpha_1, q_1, r_1, p_1]^T + F_1 + C[\xi_1, \eta_1, \zeta_1]^T \tag{4a}$$

$$\dot\phi_1 = p_1 \qquad \dot\theta_1 = q_1\cos p_0 t - r_1\sin p_0 t \tag{4b}$$

for the first approximation ($\epsilon = 0$), and

$$[\dot\beta_2, \dot\alpha_2, \dot q_2, \dot r_2, \dot p_2]^T = A[\beta_2, \alpha_2, q_2, r_2, p_2]^T + F_2 \tag{5a}$$

$$\dot\phi_2 = p_2 + q_1\theta_1\sin p_0 t + r_1\theta_1\cos p_0 t$$
$$\dot\theta_2 = q_2\cos p_0 t - r_2\sin p_0 t \tag{5b}$$

for the second approximation ($\epsilon^2 = 0$), and so on. Here

$$A = \begin{bmatrix} y_\beta & p_0 & 0 & -1 & 0 \\ -p_0 & z_\alpha & 1 & 0 & 0 \\ -m_{\dot\alpha}p_0 & \bar m_\alpha & \bar m_q & i_2 p_0 & 0 \\ n_\beta & \bar n_\alpha & \bar n_q & n_r & n_p \\ l_\beta & l_{\xi_\alpha}\xi_0 & l_q & l_r & l_p \end{bmatrix}$$

$$C=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & m_\eta & 0 \\ n_\xi & 0 & n_\zeta \\ l_\xi & 0 & l_y \end{bmatrix} \qquad F_1=\begin{bmatrix} \sin p_0 t \\ \cos p_0 t \\ m_{\dot\alpha}\cos p_0 t \\ 0 \\ 0 \end{bmatrix} \qquad F_2=\begin{bmatrix} 0 \\ 0 \\ 0 \\ n_{\xi\alpha}\xi_1\alpha_1 \\ l_{\xi\alpha}\xi_1\alpha_1+l_{\beta\alpha}\beta_1\alpha_1+l_{r\alpha}r_1\alpha_1-i_1 q_1 r_1 \end{bmatrix}$$

where $\bar{n}_\alpha = n_{\xi\alpha}\xi_0 + n_{p\alpha}p_0, \bar{n}_q = n_q - i_3 p_0$; the control parameter ξ_0 is as in Eqs. (3), while the control functions ξ_1, η_1, ζ_1, and the components $\beta_1, \alpha_1, q_1, r_1$ of the corresponding sosution of (4a) which appear in F_2, are as defined below.

Equations (4) are considered in Ref. 3 as a generating system. This set of seven equations splits into two groups. Equations (4a) constitute the first group, which is actually an independent system of five equations and – given the initial conditions and control input – completely defines five components of the solution. Assume that it admits a periodic solution: the generating solution. According to the theory underlying Poincaré's method of small parameters, this is a satisfactory approximation of the solution of the system with $\epsilon \neq 0$, for a sufficiently small ϵ, if matrix A is stable. That means that under such a condition the system with a sufficiently small ϵ admits a unique periodic solution depending on ϵ, which is asymptotically stable and tends toward the generating solution as $\epsilon \to 0$. The remaining two components of the generating solution ϕ_1 and θ_1 are obtained by quadratures according to the second group (4b). However, the zero solution of the variational system corresponding to the generating solution including ϕ and θ is neutrally stable only, and the complete generating system (4) admits a family of periodic solutions depending upon some parameters. In Ref. 3 we have shown the condition allowing us to find those values of the parameters for which the corresponding solution may actually be a first approximation of a solution of the system with $\epsilon \neq 0$. This condition, requiring that the mean value of the right-hand side of the last equation in (4) be zero, has been incorporated as an additional constraint, which – along with those deriving from the specific needs of the particular motion sought – determine the solution and the corresponding control input. Specifically, the conditions according to which rolling is steady, flight is horizontal, aircraft is not sideslipping, and its wings are level at the initial instant along with the previously mentioned condition of bifurcation, eventually lead to the following expressions of the seven state variables

$$\beta = \epsilon\,\beta_1 = -(\epsilon/y_\beta)\sin p_0 t \tag{6a}$$

$$\alpha = \epsilon\,\alpha_1 = -(\epsilon/z_\alpha)\cos p_0 t \tag{6b}$$

$$q = \epsilon q_1 = \epsilon\, p_0 h \sin p_0 t \tag{6c}$$

$$r = \epsilon\, r_1 = -\epsilon\, p_0 h \cos p_0 t \tag{6d}$$

$$p = p_0 + \epsilon\, p_1 = p_0 \tag{6c}$$

$$\phi = \phi_0 + \epsilon\,\phi_1 = p_0 t \tag{6f}$$

$$\theta = \epsilon\,\theta_1 = \epsilon(k - h\cos 2p_0 t)/2 \tag{6g}$$

where $h = -1/y_\beta + 1/z_\alpha$, $k = -1/y_\beta - 1/z_\alpha$, and the three control variables

$$\xi = \xi_0 + \epsilon\,\xi_1 = \xi_0 + \epsilon(\xi_1' \sin p_0 t + \xi_1'' \cos p_0 t)$$

$$\eta = \eta_0 + \epsilon\,\eta_1 = \epsilon(\eta_1' \sin p_0 t + \eta_1'' \cos p_0 t)$$

$$\zeta = \zeta_0 + \epsilon\,\zeta_1 = \zeta_0 + \epsilon(\zeta_1' \sin p_0 t + \zeta_1'' \cos p_0 t) \tag{7}$$

where ξ_0, η_0, and ζ_0 are as in Eqs. (3) and

$$\xi_1' = (n_\zeta l_1 - l_\zeta n_1)/\delta_1 \qquad \xi_1'' = (n_\zeta l_2 - l_\zeta n_2)/\delta_1$$

$$\eta_1' = m_1/m_\eta \qquad \eta_1'' = m_2/m_\eta$$

$$\zeta_1' = (l_\xi n_1 - n_\xi l_1)/\delta_1 \qquad \zeta_1'' = (l_\xi n_2 - n_\xi l_2)/\delta_1 \tag{7$'$}$$

with

$$l_1 = l_\beta / y_\beta - h l_q p_0 \qquad l_2 = l_{\xi\alpha}\xi_0/z_\alpha + h l_r p_0$$

$$n_1 = n_\beta / y_\beta - p_0 h(\bar{n}_q - p_0) \qquad n_2 = \bar{n}_\alpha / z_\alpha + h n_r p_0$$

$$m_1 = -p_0(\bar{m}_q - m_{\dot\alpha}/y_\beta) \qquad m_2 = m_\alpha / z_\alpha + h(1 + i_2)p_0^2$$

$$\delta_1 = l_\xi n_\zeta - n_\xi l_\zeta$$

In Figs. 1 and 2 dashed lines represent the ideal variation with time of α and θ as given in Eqs. (6).

Controls Consistent with a Motion

We have used the expression "controls consistent with a motion" in Ref. 3 to designate any control functions for which the corresponding system of equations admits a solution that describes the given motion. Controls as in (7) are consistent with the ideal steady rolling described by Eqs. (6). Two features should be emphasized.

a) No claim to a strict physical fidelity is implied. Input functions in (7) do not represent precisely what a pilot would do. Besides this is both individual and generally speaking unrepeatable. Relations in (7) are the *ideal* control functions corresponding to an ideal motion (control functions for which system (4) admits (6) as a solution). They are used in the same spirit as the fixed controls when studying stability of a steady straight flight.

b) For actually obtaining a given motion, it is not enough to use controls consistent with that motion. The corresponding solution that describes the motion should also be a stable solution of the system.

Attitude Instability

Denote

$$x_1 = \beta - \epsilon\beta_1, \quad x_2 = \alpha - \epsilon\alpha_1, \quad x_3 = q - \epsilon q_1, \quad x_4 = r - \epsilon r_1,$$

$$x_5 = p - p_0, \quad y_1 = \phi - p_0 t, \quad y_2 = \theta - \epsilon\theta_1 \tag{8a-8g}$$

Then, by taking systems (1) and (4) into account, the variational system corresponding to the ideal solution (6) will be

$$\dot{x} = (A + \epsilon D(t))x + \epsilon G(t)y + O(\epsilon^2) \tag{9a}$$

$$\dot{y} = \epsilon H(t)y + K(t)x + O(\epsilon^2) \tag{9b}$$

where A is as in Eq. (2) while

$$D=\begin{bmatrix} 0 & 0 & 0 & 0 & \alpha_1 \\ 0 & 0 & 0 & 0 & -\beta_1 \\ 0 & 0 & 0 & 0 & -m_{\dot\alpha}\beta_1+i_2r_1 \\ 0 & n_{\xi\alpha}\xi_1 & 0 & 0 & n_{p\alpha}\alpha_1-i_3q_1 \\ 0 & l_{\beta\alpha}\beta_1+l_{r\alpha}r_1+l_{\xi\alpha}\xi_1 & -i_1r_1 & l_{r\alpha}\alpha_1-i_1q_1 & 0 \end{bmatrix}$$

$$G=\begin{bmatrix} \cos p_0t & 0 \\ -\sin p_0t & \theta_1\cos p_0t \\ -m_{\dot\alpha}\sin p_0t & m_{\dot\alpha}\theta_1\cos p_0t \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad H=\begin{bmatrix} 0 & -p_0h\cos 2p_0t \\ p_0h\cos 2p_0t & 0 \end{bmatrix}$$

$$K=\begin{bmatrix} 0 & 0 & 0 & \epsilon\theta_1\cos p_0t & 1 \\ 0 & 0 & \cos p_0t & -\sin p_0t & 0 \end{bmatrix}$$

with β_1, α_1, q_1, and r_1 as in (6) and ξ_1 as in (7)′.

An examination of system (9) might answer the question why controls consistent with an ideal steady rolling within an approximation of order ϵ, are sufficient to achieve the maneuver acceptably for certain values of p_0 (Fig. 1) and result in a dangerous unstable motion for other values (Fig. 2). The form of system (9) (parameter-excited) suggests that parametric resonance[5] is responsible for this anomaly. However, a complete analysis of system (9) that would allow us to infer parametric resonance and determine the resonant values of p_0 is exceedingly involved.§ Therefore only a simplified study of system (9) will be presented in the following.

Simplified Equations of Attitude Deviation

Assume that the stability margin of matrix A is large enough to cause an acceptable rate of decay of the x variables. Group (9b) will be considered as a separate system in y and the terms in x as perturbations. The simplified system in y can then be written in the following form¶

$$\delta\dot\phi=(1/2)\epsilon^2p_0h\sin 2p_0t(k-h\cos 2p_0t)\delta\phi-\epsilon p_0h\cos 2p_0t\,\delta\theta \tag{10a}$$

$$\delta\dot\theta=\epsilon\, p_0h\cos 2p_0t\,\delta\phi \tag{10b}$$

Let $C(t,\epsilon)$ be the fundamental matrix of solutions of system (10) corresponding to the initial value $C(0,\epsilon)=I$. We may consider its power series expansion in ϵ. Hence, successive approximations to the characteristic equation of system (10) can be calculated according to a pattern described in the Appendix (see Chap. 3 of Ref. 5 or Section 60 of Ref. 6).

Let $C(t,\epsilon)=C_0(t)+\epsilon C_1(t)+\epsilon^2C_2(t)+\epsilon^3C_3(t)+\ldots$ be the fundamental solution matrix, with $C_0(0)=I$, $C_i=0$, $i=1,2,3,\ldots$ Then we can successively calculate

$$C_0(t)=I; \qquad C_1(t)=\begin{bmatrix} 0 & -(h/2)\sin 2p_0t \\ (h/2)\sin 2p_0t & 0 \end{bmatrix}$$

$$C_2(t)=\begin{bmatrix} -h^2(1-\cos 4p_0t)/8+hk(1-\cos 2p_0t)/4 & 0 \\ 0 & -(h/4)^2(1-\cos 4p_0t) \end{bmatrix}$$

$$C_3(t)=\begin{bmatrix} 0 & h^2[-p_0kt-(h/8)\sin p_0t+(k/4)\sin 4p_0t+(h/24)\sin 6p_0t]/8 \\ h^2[-p_0kt+(k-h/4)\sin 2p_0t-(k/4)\sin 4p_0t+(h/12)\sin 6p_0t]/8 & 0 \end{bmatrix}$$

...

and still further the successive approximations to the characteristic equation $\det[\lambda I-C(T,\epsilon)]=0$, where the period $T=\pi/p_0$

$$E_0(\lambda,\epsilon)\equiv\det(\lambda I-C_0)\equiv\begin{vmatrix} \lambda-1 & 0 \\ 0 & \lambda-1 \end{vmatrix}=0$$

$$E_1(\lambda,\epsilon)\equiv\det[\lambda I-C_0-\epsilon C_1(T)]\equiv\begin{vmatrix} \lambda-1 & 0 \\ 0 & \lambda-1 \end{vmatrix}=0$$

$$E_2(\lambda,\epsilon)\equiv\det[\lambda I-C_0-\epsilon C_1(T)-\epsilon^2C_2(T)]$$

$$\equiv\begin{vmatrix} \lambda-1 & 0 \\ 0 & \lambda-1 \end{vmatrix}=0$$

$$E_3(\lambda,\epsilon)\equiv\det[\lambda I-C_0-\epsilon C_1(T)-\epsilon^2C_2(T)-\epsilon^3C_3(T)]$$

$$\equiv\begin{vmatrix} \lambda-1 & \epsilon^2\pi h^2k/8 \\ \epsilon^2\pi h^2k/8 & \lambda-1 \end{vmatrix}=0$$

...

The approximation of order ϵ^3 is the first to give a nonambiguous answer as one of the roots of $E_3(\lambda,\epsilon)=0$, viz. $\lambda=1+\epsilon^2\pi h^2k/8$ with $k>0$ is in the instability region (in the exterior of the unit circle: $|\lambda|=1$), though in rather a close neighborhood of the boundary. The preceding approximations have their characteristic roots on the very boundary (both roots equal to 1), and therefore are inconclusive.

§As far as I know, the problem of parametric resonance has been solved completely only for systems of some particular forms (e.g., for the canonical equations, see Ref. 5, Chap. 5).

¶We consider the second approximation in ϵ, rather than the first one, as the latter is not conclusive because it gives only neutral stability. If the term of order ϵ^2 in (10) is dropped, $\delta\phi\delta\dot\phi+\delta\theta\delta\dot\theta$ vanishes, hence $\delta\phi^2+\delta\theta^2=\delta\phi^2(0)+\delta\theta^2(0)=\text{const}$.

It is noteworthy that the unstable root $\lambda = 1 + \epsilon^3 \pi (1/z_\alpha - 1/y_\beta)^2 (-1/z_\alpha - 1/y_\beta)/8$ (where y_β and z_α are negative) is independent of the roll rate p_0, and depends on the flight condition and the aircraft characteristics, through the agency of the lift-curve and side-force-curve slopes (z_α and y_β), in a straightforward way.

Consequently, steady rolling is always unstable with respect to attitude of the aircraft, and it is the more so the lower the value of $-z_\alpha$ and $-y_\beta$, particularly so in case of y_β.

A more detailed analysis may emphasize other influences as well. The es ential conclusion of the above simplified discussion is, however, the existence of an instability in steady rolling due to the position angles without regard to the magnitude of rolling velocity.

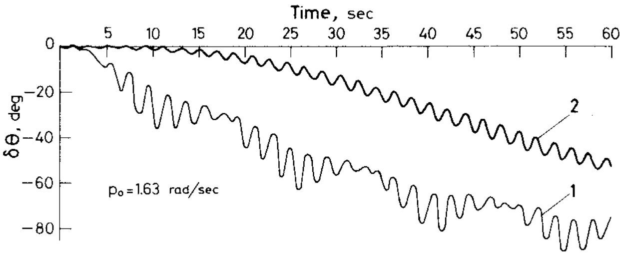

Fig. 3 Time histories of the deviation $\delta\theta$ of the elevation angle of x principal axis from its steady-state periodic variation during 60 sec at $p_0 = 1.63$ and 3.84 rad/sec for the original aircraft (1) and a fictitious augmented aircraft (2).

Nonsimplified System

To more accurately estimate the rate of growth due to instability, we have integrated the system of the nonsimplified seven equations in deviations (8) over an interval of 60 sec for the initial condition zero (at the same couple of values of p_0 as noted previously). The accuracy of available controls is still assumed of order ϵ (it would seem rather unrealistic to assume any greater accuracy) so that the system in deviations does not include control terms. It can readily be obtained from Eqs. (1, 4, 6, and 7), and may be written as

$$\delta\dot{\beta} = y_\beta \delta\beta + p_0 \delta\alpha - \delta r + \epsilon\alpha_1(t)\delta p + \delta\alpha\delta p + \epsilon[\sin(\delta\phi + p_0 t)\cos(\delta\theta + \epsilon\theta_1(t)) - \sin p_0 t]$$

$$\delta\dot{\alpha} = -p_0\delta\beta + z_\alpha\delta\alpha + \delta q - \epsilon\beta_1(t)\delta p - \delta\beta\delta p + \epsilon[\cos(\delta\phi + p_0 t)\cos(\delta\theta + \epsilon\theta_1(t)) - \cos p_0 t]$$

$$\delta\dot{q} = -m_{\dot{\alpha}} p_0 \delta\beta + \bar{m}_\alpha \delta\alpha + \bar{m}_q \delta q + i_2 p_0 \delta r + \epsilon(i_2 r_1(t) - m_{\dot{\alpha}}\beta_1(t))\delta p - m_{\dot{\alpha}}\delta\beta\delta p$$
$$+ i_2 \delta r \delta p + \epsilon[m_{\dot{\alpha}}\cos(\delta\phi + p_0 t)\cos(\delta\theta + \epsilon\theta_1) - m_{\dot{\alpha}}\cos p_0 t]$$

$$\delta\dot{r} = n_\beta\delta\beta + (\bar{n}_\alpha + \epsilon n_{\xi\alpha}\xi_1(t))\delta\alpha + n_q\delta q + n_r\delta r + [n_p + \epsilon(n_{p\alpha}\alpha_1(t) - i_3 q_1(t))]\delta p$$
$$+ n_{p\alpha}\delta\alpha\delta p - i_3\delta q\delta p + \epsilon^2 n_{\xi\alpha}\xi_1(t)\alpha_1(t)$$

$$\delta\dot{p} = (l_\beta + \epsilon l_{\beta\alpha}\alpha_1(t))\delta\beta + [l_{\xi\alpha}\xi_0 + \epsilon(l_{\xi\alpha}\xi_1(t) + l_{\beta\alpha}\beta_1(t) + l_{r\alpha}r_1(t))]\delta\alpha + (l_q$$
$$- \epsilon i_1 r_1(t))\delta q + [l_r + \epsilon(l_{r\alpha}\alpha_1(t) - i_1 q_1(t))]\delta r + l_p\delta p + l_{\beta\alpha}\delta\beta\delta\alpha$$
$$+ l_{r\alpha}\delta\alpha\delta r - i_1\delta q\delta r + \epsilon^2[l_{\beta\alpha}\beta_1(t)\alpha_1(t) + l_{r\alpha}\alpha_1(t)r_1(t) - i_1 q_1(t)r_1(t) + l_{\xi\alpha}\xi_1(t)\alpha_1(t)]$$

$$\delta\dot{\phi} = \delta p + [(\delta q + \epsilon q_1(t))\sin(\delta\phi + p_0 t) + (\delta r + \epsilon r_1(t))\cos(\delta\phi + p_0 t)]\tan(\delta\theta + \epsilon\theta_1(t))$$

$$\delta\dot{\theta} = [\delta q + \epsilon q_1(t)]\cos(\delta\phi + p_0 t) - (\delta r + \epsilon r_1(t))\sin(\delta\phi + p_0 t) - \epsilon[q_1(t)\cos p_0 t - r_1(t)\sin p_0 t] \quad (11)$$

We have plotted in Fig. 3 the time histories of the deviation $\delta\theta$ of the elevation angle from its steady-state periodic variation [as in Eq. (6)], for $p_0 = 1.63$ and 3.84 rad/sec, during 60 sec. The attitude instability predicted previously accounts for the (rather slightly) diverging trend of the response. Some other perturbations are obviously superimposed.

To make the influence of the force derivatives y_β and z_α more conspicuous, we now assume that they can be increased by a factor, say, 1.5, while the rest of the system is left unchanged. The resulting variation of $\delta\theta$ is also represented for comparison in the same figure. Instability is obviously lessened.

Though generally gentle, attitude instability increases the receptivity of the response to various perturbing causes. One of such causes, perhaps the most significant one since it appears in the early stages of a rolling maneuver, is the roll-resonance effect. As will subsequently be shown, this effect may be quite striking although it is only of second-order with respect to ϵ. It will also be shown that the effect is weaker, to the point of virtual absence, when attitude instability is diminished (by an increase of $-y_\beta$ and $-z_\alpha$).

The simplified analysis just presented has enabled us to infer the instability for system (9) and point out the role of y_β and z_α in controlling the extent of this instability. However, it is of no avail in the determination of the resonant values of p_0. The following equally simplified study helps us find such values.

Second-Order Approximation

Vector F_2 in Eq. (5a) is now defined by means of relations (6) and (7)′ (which determine the functions β_1, α_1, q_1, r_1, and ξ_1) and we can compute the solution in second approximation with respect to ϵ. We have

$$F_2 = \mathfrak{F}_0 + \mathfrak{F}_1 \sin 2p_0 t + \mathfrak{F}_2 \cos 2p_0 t$$

where

$$\mathfrak{F}_0 = \mathfrak{F}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -n_{\xi\alpha}\xi_1''/2z_\alpha \\ (p_0 h l_{r\alpha} - l_{\xi\alpha}\xi_1'')/2z_\alpha \end{bmatrix}$$

$$\mathfrak{F}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -n_{\xi\alpha}\xi_1'/2z_\alpha \\ -[i_1 p_0^2 h^2 + (l_{\xi\alpha}\xi_1' - l_{\beta\alpha}/y_\beta)/z_\alpha]/2 \end{bmatrix}$$

Then β_2, α_2, q_2, r_2, and p_2 in Eq. (2) will be determined in the following form

$$\beta_2 = \beta_{20} + \beta_{21} \sin 2p_0 t + \beta_{22} \cos 2p_0 t$$

$$\alpha_2 = \alpha_{20} + \alpha_{21} \sin 2p_0 t + \alpha_{22} \cos 2p_0 t$$

$$q_2 = q_{20} + q_{21} \sin 2p_0 t + q_{22} \cos 2p_0 t$$

$$r_2 = r_{20} + r_{21} \sin 2p_0 t + r_{22} \cos 2p_0 t$$

$$p_2 = p_{20} + p_{21} \sin 2p_0 t + p_{22} \cos 2p_0 t$$

by a linear set of five algebraic equations, namely

$$A\,[\beta_{20}, \alpha_{20}, q_{20}, r_{20}, p_{20}]^T = -\mathfrak{F}_0$$

and a linear set of ten equations, namely

$$A\,[\beta_{21}, \alpha_{21}, q_{21}, r_{21}, p_{21}]^T$$

$$+ 2p_0\,[\beta_{22}, \alpha_{22}, q_{22}, r_{22}, p_{22}]^T = -\mathfrak{F}_1$$

$$- 2p_0\,[\beta_{21}, \alpha_{21}, q_{21}, r_{21}, p_{21}]^T$$

$$+ A\,[\beta_{22}, \alpha_{22}, q_{22}, r_{22}, p_{22}]^T = -\mathfrak{F}_2$$

Transient Component and Roll Resonance

Consequently, we have the following initial values of the coefficients of ϵ^2 in the steady-state solution: $\beta_2(0) = \beta_{20} + \beta_{22}$, $\alpha_2(0) = \alpha_{20} + \alpha_{22}$, $q_2(0) = q_{20} + q_{22}$, $r_2(0) = r_{20} + r_{22}$, $p_2(0) = p_{20} + p_{22}$. They generally differ from zero. Now we have computed both the time histories of the state variables of system (1) and the respective components of the generating ideal solution (6) for the same initial conditions. This implies that, in addition to the steady-state components calculated as shown in the preceding section, the second-approximation correction also includes a transient component described (in a linear approximation) by the solution of the variational system

$$[\Delta\dot\beta, \Delta\dot\alpha, \Delta\dot q, \Delta\dot r, \Delta\dot p]^T = A\,[\Delta\beta, \Delta\alpha, \Delta q, \Delta r, \Delta p]^T$$

corresponding to the following initial conditions:

$$\Delta\beta(0) = -\epsilon^2\beta_2(0), \quad \Delta\alpha(0) = -\epsilon^2\alpha_2(0),$$

$$\Delta q(0) = -\epsilon^2 q_2(0), \quad \Delta r(0) = -\epsilon^2 r_2(0),$$

$$\Delta p(0) = -\epsilon^2 p_2(0)$$

Let

$$[\Delta\beta(t), \ \Delta\alpha(t), \ \Delta q(t), \ \Delta r(t), \ \Delta p(t)]^T$$

be this solution. Then the first five components of the complete solution in second approximation as regard the small parameter ϵ of system (1) is given by

$$\epsilon\beta_1 + \epsilon^2\beta_2 + \Delta\beta = \epsilon\beta_1 + \epsilon^2\bar\beta_2$$

$$\epsilon\alpha_1 + \epsilon^2\alpha_2 + \Delta\alpha = \epsilon\alpha_1 + \epsilon^2\bar\alpha_2$$

$$\epsilon q_1 + \epsilon^2 q_2 + \Delta q = \epsilon q_1 + \epsilon^2\bar q_2$$

$$\epsilon r_1 + \epsilon^2 r_2 + \Delta r = \epsilon r_1 + \epsilon^2\bar r_2$$

$$p_0 + \epsilon^2 p_2 + \Delta p = p_0 + \epsilon^2\bar p_2$$

The remaining two components, namely ϕ and θ, are obtained by quadratures according to Eqs. (4b) and (5b). In the latter, $\bar p_2$, $\bar q_2$, and $\bar r_2$ are substituted for p_2, q_2, and r_2, respectively.

To be definitive, let us consider the component $\bar\theta_2$. It will include steady-state and transient parts, the latter due to Δq and Δr. The steady-state part, viz.

$$\theta_2 = \theta_2(0) + (1/p_0)\,[q_{20} + (q_{22} - r_{21})/2]\ \sin p_0 t$$

$$+ (1/p_0)\,[r_{20} - (q_{21} + r_{22})/2]\ \cos p_0 t$$

$$+ (1/6p_0)\,(q_{22} + r_{21})\ \sin 3p_0 t$$

$$+ (1/6p_0)\,(r_{22} - q_{21})\ \cos 3p_0 t$$

multiplied by ϵ^2 is usually negligible as compared to $\epsilon\theta_1$. Matrix A in typical situations has at least one pair of complex eigenvalues, say $\sigma \pm i\omega$. Then the expression for the transient part $\Delta\theta_2$ has $[\sigma^2 + (\omega - p_0)^2]^{1/2}$ in the denominator, and therefore is the potential source of a resonance phenomenon for p_0 close to ω when $|\sigma|$ is small.

Resonance is a property of linear systems. We can therefore expect high peak values of the solution of the linear system in deviations to occur in an early stage of the maneuver in resonant or quasiresonant situations (i.e., when p_0 is equal or close to a natural frequency of the variational system when the corresponding damping is low). The nonlinear terms in such situations are not negligible, and come to prevail soon after the start. Hence, near resonance 'linear response' cannot offer any true description of the actual motion either qualitatively or quantitatively. However, it may successfully serve as a model in far-from-resonance conditions, over an initial interval of time long enough to be useful for practical purposes. We have plotted for illustration in Fig. 4 the variation with time of θ in linear approximation, viz. $\epsilon\theta_1 + \delta\theta_{\text{linear}}$, with $\delta\theta_{\text{linear}}$ calculated by integrating the following linear system in deviations [linear part of Eq. (11)].

$$\begin{bmatrix} \delta\dot\beta \\ \delta\dot\alpha \\ \delta\dot q \\ \delta\dot r \\ \delta\dot p \\ \delta\dot\phi \\ \delta\dot\theta \end{bmatrix} = \left[\begin{matrix} y_\beta & p_0 & 0 & -1 & \epsilon\alpha_1 \\ -p_0 & z_\alpha & 1 & 0 & -\epsilon\beta_1 \\ -m_{\dot\alpha}p_0 & \bar m_\alpha & \bar m_q & i_2 p_0 & \epsilon(i_2 r_1 - m_{\dot\alpha}\beta_1) \\ n_\beta & \bar n_\alpha - \epsilon n_{\xi\alpha}\xi_1 & n_q - i_3 p_0 & n_r & n_p + \epsilon(n_{p\alpha}\alpha_1 - i_3 q_1) \\ l_\beta + \epsilon l_{\beta\alpha}\alpha_1 & l_{\xi\alpha}\xi_0 + \epsilon(l_{\xi\alpha}\xi_1 + l_{\beta\alpha}\beta_1 + l_{r\alpha}r_1) & l_q - \epsilon i_1 r_1 & l_r + \epsilon(l_{r\alpha}\alpha_1 - i_1 q_1) & l_p \\ 0 & 0 & \epsilon\theta_1 \sin p_0 t & \epsilon\theta_1 \cos p_0 t & 1 \\ 0 & 0 & \cos p_0 t & -\sin p_0 t & 0 \end{matrix}\right.$$

$$\left.\begin{matrix} \epsilon\cos p_0 t & -\epsilon^2\theta_1 \sin p_0 t \\ -\epsilon\sin p_0 t & -\epsilon^2\theta_1 \cos p_0 t \\ -\epsilon m_{\dot\alpha}\sin p_0 t & -\epsilon^2 m_{\dot\alpha}\theta_1 \cos p_0 t \\ 0 & 0 \\ 0 & 0 \\ \epsilon^2\theta_1 (q_1 \cos p_0 t - r_1 \sin p_0 t) & \epsilon(q_1 \sin p_0 t + r_1 \cos p_0 t) \\ -\epsilon(q_1 \sin p_0 t + r_1 \cos p_0 t & 0 \end{matrix}\right]$$

$$\times \begin{bmatrix} \delta\beta \\ \delta\alpha \\ \delta q \\ \delta r \\ \delta p \\ \delta\phi \\ \delta\theta \end{bmatrix} + \epsilon^2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ n_{\xi\alpha}\xi_I\alpha_I \\ l_{\xi\alpha}\xi_I\alpha_I + l_{\beta\alpha}\beta_I\alpha_I + l_{r\alpha}\alpha_I r_I - i_I q_I r_I \\ \theta_I\,(q_I \sin p_0 t + r_I \cos p_0 t) \\ 0 \end{bmatrix} \qquad (12)$$

with $\beta_I,\alpha_I,q_I,r_I,\theta_I$ as in Eq. (6), ξ_0 and ξ_I as in Eqs. (3) and (7)′, respectively, integration being carried out for zero initial conditions. The curve obtained for $p_0 = 1.63$ rad/sec is very much unlike the corresponding one in Fig. 2, while for $p_0 = 3.84$ rad/sec resemblance to that in Fig. 1 is quite remarkable.

As can be seen from Fig. 5, 1.63 rad/sec is a resonant value of p_0 (the only one that is of practical interest for the given aircraft and flight condition). There is a range about this value, somewhere between 1.5 and 2.2 rad/sec, where the elevation angle exhibits large deviations. As a matter of fact this whole range is situated in the domain supposed to be stable according to the conventional linear analysis (where det $A > 0$). A few qualitative comments concerning the main parameters influencing the resonant values of p_0, based on a simplified analysis, are given in the last section.

Influence of the Side-Force Curve and the Lift Curve Slopes: Illustrative Example

We have pointed out that a) attitude instability increases the sensitivity of the response to disturbances, particularly the resonance effect, and b) the rate of growth caused by instability mainly depends on the value of y_β and z_α.

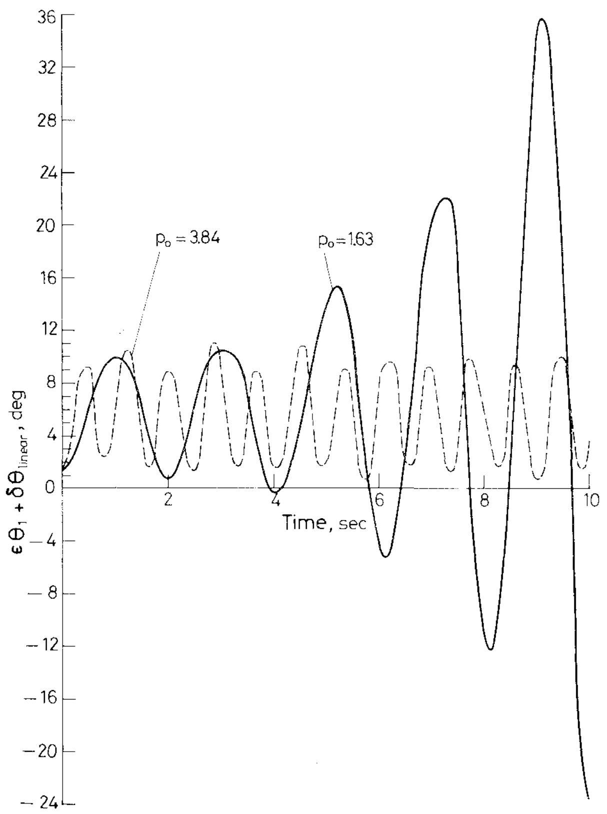

Fig. 4 Variation of the pitch angle with time in linear approximation: $\theta = \epsilon\theta_I + \delta\theta_{\text{linear}}$, with $\delta\theta_{\text{linear}}$ calculated according to Eq. (12).

Assume for illustration that y_β (Case 1) and then both y_β and z_α (Case 2) are augmented by the factor 1.5., while the rest of the parameters are left unchanged in the numerical example already used. Attitude variation is virtually trimmed for at least the first ten seconds at $p_0 = 1.63$ rad/sec (see Fig. 6 as compared to Fig. 2). Now change in values of y_β and z_α resultes in a shift of the resonance values of p_0 (see Fig. 7). The imaginary part of the lower frequency mode ω_I equals p_0 when $p_0 = 1.38$ and 2.00 rad/sec in Case 1, and $p_0 = 1.41$ and 2.00 rad/sec in Case. 2. The real parts corresponding to $p_0 = 2$

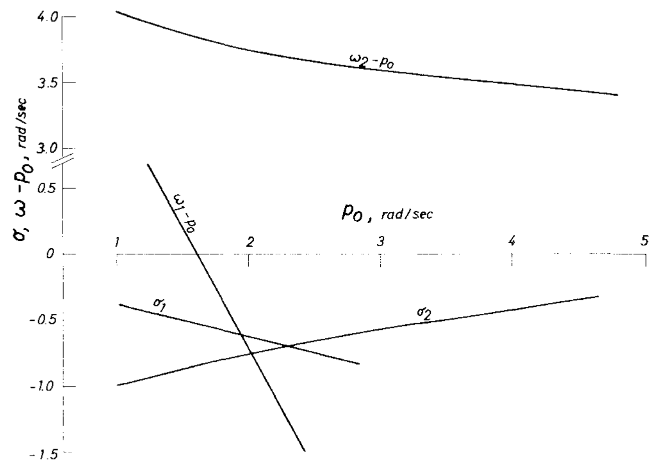

Fig. 5 Determination of the resonant value of p_0, and variation of the dissipation factor σ with p_0.

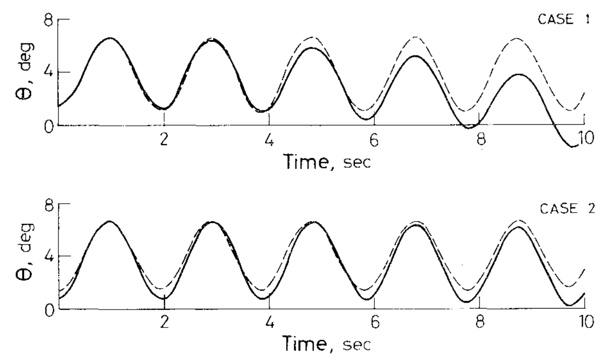

Fig. 6 Variation of θ during 10 sec at $p_0 = 1.63$ rad/sec when y_β (Case 1) and simultaneously both y_β and z_α (Case 2) are increased by the factor 1.5 as compared to the respective ideal periodic variations.

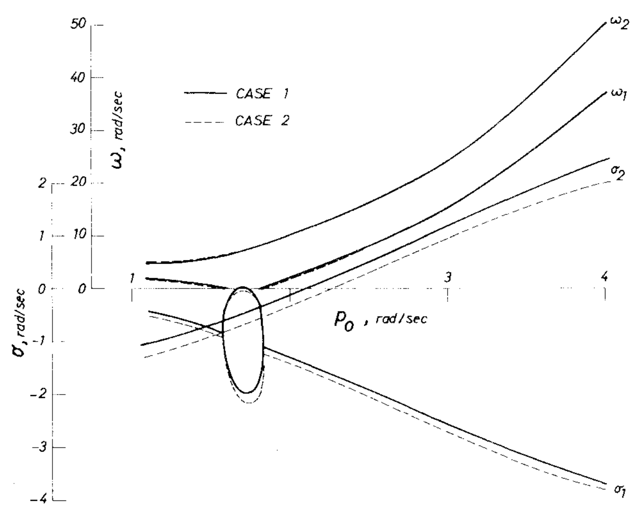

Fig. 7 Real and imaginary parts of characteristic roots vs p_0 when y_β (Case 1) and both y_β and z_α (Case 2) are augmented by the factor 1.5.

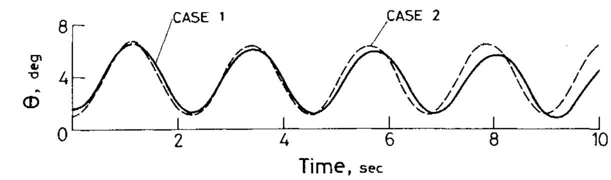

Fig. 8 Variation of θ during 10 sec at the 'resonant values' of p_0 for both cases of (fictitiously) augmented aircraft: for Case 1, $p_0 = 1.38$, for Case 2, $p_0 = 1.41$ rad/sec.

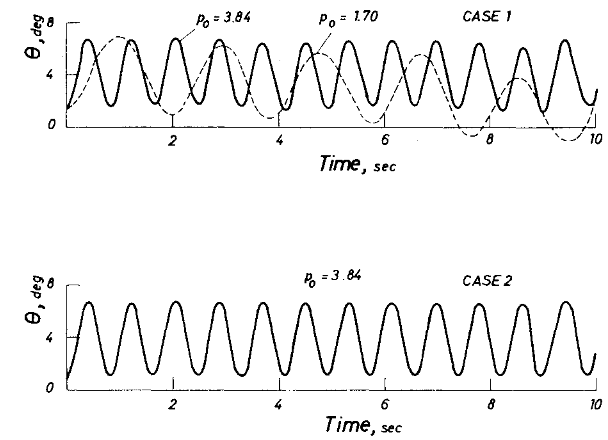

Fig. 9 Variation of θ during 10 sec at a divergence value of p_0 in Case 1 ($p_0 = 1.7$ rad/sec, $\lambda = +.032$ sec^{-1}), and at an oscillatory-instability value in both cases ($p_0 = 3.84$ rad/sec: $\sigma_2 = 2.12$ in Case 1, and 1.86 sec^{-1} in Case 2).

rad/sec have relatively high values in both cases, hence 1.38 and 1.41 are the only values of p_0 to be considered. As can be seen from Fig. 8, the resonance effect is practically speaking suppressed and the attitude variation remains acceptably close to the ideal periodic variation.

Augmenting y_β and z_α affects the spectrum of matrix A in some other ways as well (Fig. 7). Thus the lower frequency oscillatory mode splits into two aperiodic modes for a certain range of p_0 (from about 1.6-1.8 rad/sec). Within this range, between approximately 1.65 and 1.75 rad/sec matrix A has one eigenvalue which is nearly zero. (In Case 1 in the vicinity of $p_0 = 1.7$ one of the real roots is slightly positive). Then again when p_0 exceeds a certain value (of about 2.18 in Case 1, and 2.37 in Case 2) the higher frequency oscillations become unstable. Both these secondary effects are, however, negligible during any initial interval of time of practical interest. Thus, as is seen in Fig. 9, the positive real root at $p_0 = 1.7$ rad/sec in Case 1 ($\lambda = +0.032$ sec^{-1}) proves to be rather harmless during 8-10 sec. When $p_0 = 3.84$ rad/sec no deviation of irregularity is discernible in either case during at least ten seconds, although at this value of the roll rate, matrix A is unstable, admitting complex eigenvalues with positive real parts ($\sigma_2 = 2.12$ and 1.86 sec^{-1} for Cases 1 and 2, respectively).

Summing up, low values of y_β and z_α account for the unpleasant behavior in roll near resonance. Now the specifications that usually define y_β and z_α do not include one to prevent roll resonance. (Moreover performance and resonance requirements as regard z_α would be conflicting). Therefore, the most natural way of avoiding an unwanted resonance effect is to shift the resonance range of p_0 beyond the aircraft maximum roll rate capability or to place operating restrictions on the maximum allowable roll rate for a given aircraft and flight condition.

Influence of Static Stability

A simplified discussion may help us discern those parameters that control the imaginary parts of the characteristic roots of matrix A, and hence are essential in defining the resonant values of the rolling velocity. First, consider the case of two pairs of complex eigenvalues of matrix B (obtained from A by removing both the fifth row and fifth column under the assumption either that n_p is negligible or that rolling velocity can be maintained strictly constant). Let $\sigma_1 \pm i\omega_1$ and $\sigma_2 \pm i\omega_2, \omega_1 < \omega_2$, be these eigenvalues. Then disregarding some small terms in the identity

$$(\lambda^2 - 2\sigma_1\lambda + \sigma_1^2 + \omega_1^2)(\lambda^2 - 2\sigma_2\lambda + \sigma_2^2 + \omega_2^2) = \det(B - \lambda I)$$

leads to the following relations

$$\sigma_1 + \sigma_2 = (y_\beta + z_\alpha + \bar{m}_q + n_r)/2$$

$$\omega_1^2 + \omega_2^2 + 4\sigma_1\sigma_2\omega_1\omega_2 = -(\sigma_1^2 + \sigma_2^2) - \bar{m}_\alpha + n_\beta + (1 - i_2 i_3)p_0^2 + \bar{m}_q n_r + z_\alpha n_r + z_\alpha \bar{m}_q + y_\beta n_r + y_\beta z_\alpha \quad (13)$$

$$(\omega_1^2 + \sigma_1^2)(\omega_2^2 + \sigma_2^2) = \det B \quad (14)$$

The damping derivatives $\bar{m}_q$, n_r, y_β, and z_α are small as compared to the static stability derivatives $|\bar{m}_\alpha|$ and n_β, hence σ_1 and σ_2 are relatively small.[1-3] To establish the main influences, only the main terms are left now in Eqs. (13) and (14). Then the following approximate relations

$$\omega_1^2 + \omega_2^2 = -\bar{m}_\alpha + n_\beta + (1 - i_2 i_3)p_0^2 \equiv 2b$$

$$\omega_1^2\omega_2^2 = i_2 i_3 (p_0^2 + \bar{m}_\alpha / i_2)(p_0^2 - n_\beta / i_3) \equiv c \quad (15)$$

are readily obtained, when $\omega_1^2 = b - (b^2 - c)^{1/2}$ and $\omega_2^2 = b + (b^2 - c)^{1/2}$.

When only two of the eigenvalues of B are complex, say $\sigma \pm i\omega$, we have $\lambda_1\lambda_2 + \omega^2 = 2b$ and $\lambda_1\lambda_2\omega^2 = c$ rather than (15) (λ_1 and λ_2 are the real characteristic roots), and $\omega^2 = b + (b^2 - c)^{1/2}$, $\lambda_1\lambda_2 = b - (b^2 - c)^{1/2}$.

To avoid resonance at low values of p_0, b and c, $|m_\alpha|$ and n_β should be large. To estimate how large their values should be, using simplified systems such as the previous is not recommended. They are, however, suitable for drawing some simple, qualitative conclusions. It appears indeed that Phillips' early results[1] entail several valid qualitative conclusions if applied to finite-time rather than asymptotic behavior (viz. stability) of the aircraft in a rolling maneuver. Specifically, Phillips' intuition proves correct in as far as it implies that a) the decisive parameters controlling the response characteristics are the static stability derivatives (m_α and n_β), and b) there is some range of the critical roll-rate values for which behavior of an aircraft is unsatisfactory, and these values are higher the greater $|m_\alpha|$ and n_β. The critical roll-rate values, however, can be located in the neighborhood of the imaginary parts of the relatively lightly damped characteristic roots as well as in the divergence range (values of p_0 for which matrix B has one positive real eigenvalue, i.e., when $\det B < 0$). The former type of critical roll-rate value is perhaps of even greater interest than the latter type because of its common occurrence *plus* its rapidly developing pathological motion. Moreover, a negative free term of the characteristic polynomial of the linear fourth order system ($\det B < 0$), although detrimental in the long run, may be (and often actually *is*) beneficial in the early stages of the motion since it eliminates the oscillatory mode with the lower natural frequency and thus delays the appearance of the resonance effect until higher roll-rate values are reached or it prevents resonance altogether. It is sometimes desirable indeed to have the lower-frequency oscillatory mode split into two aperiodic modes especially as sensitivity to a change in static stability of the resonant value of p_0 often is lower than it might be expected. Thus, this value increases by 20% only (from 1.63-1.96 rad/sec) in the numerical example previously used, when both m_α and n_β are augmented by a factor as large as six.

Appendix: Some Properties of Linear Systems with Periodic Coefficients Depending on a Small Parameter

Characteristic Equation of Linear Periodic System

Consider the system

$$\dot{x} = P(t)x \tag{A1}$$

where x is an n column-vector, and $P(t)$ a periodic matrix-function with period T. Let $C(t)$ be a fundamental matrix of solutions of (A1), that means, since $C(t+T)$ is again a solution of system (A1), that

$$C(t+T) = C(t)\Lambda$$

where Λ is a constant $n \times n$ matrix. The equation in λ det $(\lambda I - \Lambda) = 0$ is referred to as the characteristic equation of system (A1). It is independent of the choice of the fundamental solution matrix (see Ref. 5, Chap. 3, Sec. 2). Consider the particular fundamental solution defined by the initial condition $C(0) = I$. Then $\Lambda = C(T)$, and the characteristic equation can be written as

$$\det\, [\lambda I - C(T)] = 0$$

Fundamental Solution Matrix as Power Series in the Small Parameter

Suppose that the right-hand side of (A1) depends on a small parameter ϵ and system (A1) can be written in the form

$$\dot{x} = [P_0 + \epsilon\, P_1(t) + \epsilon^2 P_2(t) + \ldots]x \tag{A2}$$

Then we can consider the expansion

$$C(t,\epsilon) = C_0(t) + \epsilon C_1(t) + \epsilon^2 C_2(t) + \epsilon^3 C_3(t) + \ldots$$

of the fundamental solution matrix defined by $C(0,\epsilon) = I$ and successively calculate the coefficients $C_i(t)$. Equating coefficients of equal powers of ϵ on both sides in

$$\sum_{i=0}^{\infty} \epsilon^i \dot{C}_i(t) = \left[\sum_{i=0}^{\infty} \epsilon^i P_i(t)\right] \times \left[\sum_{i=0}^{\infty} \epsilon^i C_i(t)\right]$$

leads indeed to the equations

$$\dot{C}_0(t) = P_0 C_0(t)$$

$$\dot{C}_1(t) = P_0 C_1(t) + P_1(t) C_0(t)$$

$$\dot{C}_2(t) = P_0 C_2(t) + P_1(t) C_1(t) + P_2(t) C_0(t)$$

...

which can be integrated successively for set initial conditions. In particular, for $C(0,\epsilon) = I$, i.e., $C_0(0) = I$, $C_i(0) = 0$, $i = 1,2,3,\ldots$ the formula of variation of constants gives

$$C_0(t) = e^{P_0 t}$$

$$C_\nu(t) = \int_0^t e^{P_0(t-s)} \left[\sum_{i=1}^{\nu} P_i(s) C_{\nu-i}(s)\right] ds, \quad \nu = 1,2,3,\ldots$$

Stability Criterion

Consequently, the coefficients of the characteristic equation

$$\det\, [\lambda I - C(T,\epsilon)] = 0 \tag{A3}$$

can be determined approximately by calculating $C_0(T)$, $C_1(T)$, $C_2(T)$ and so on.

The characteristic equation (A3) provides information concerning stability of the trivial solution of system (A2). Asymptotic stability means that all characteristic roots are strictly inside the unit circle, $|\lambda_i| < 1$, $i = 1,2,\ldots n$; neutral stability that they are within and/or on its boundary, $|\lambda_i| \leq 1$, and only simple elementary divisors correspond to the characteristic roots of absolute value 1; finally instability either means the existence of a root on the boundary with a multiple elementary divisor while $|\lambda_i| \leq 1$ (general solution includes secular terms) or the existence of a root in the outside of the unit circle, $|\lambda_i| > 1$ (exponential instability). Successive approximations should be continued until a clear-cut conclusion is reached, i.e., either asymptotic stability or instability is obtained: neutral stability in first approximation is inclusive. (So are the first and second approximations to the characteristic equation of system (10), having double critical roots with two simple elementary divisors; the approximation of order ϵ^3 admits, however, roots outside the unit circle, hence the trivial solution of (10) is exponentially unstable.)

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